① Structure theorem of Compact Lie algebra

Thm (i) $\mathfrak{g}$ is a compact Lie algebra $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus [\mathfrak{g}\ \mathfrak{g}]$

$[\mathfrak{g}, \mathfrak{g}]$ is semi simple.

(ii) A Simple Lie algebra $\mathfrak{g}$ is compact iff $B < 0$.

Here we assume $\mathfrak{g}$ is real. The reason is any compact complex Lie group must be a torus.

Defn: $\mathfrak{g}$ is called compact if $\exists$ G compact $\mathrm{Lie}(G) = \mathfrak{g}$.

Pf: The key is that on $\mathfrak{g}$ $\exists$ Ad invariant inner product $\langle\ ,\ \rangle$.

We define

$$\langle X, Y \rangle := \int_{M=G} \langle Ad(g)(X), Ad(g)(Y) \rangle_0 \, d\mu$$

Haar measure

$$\Rightarrow \langle Ad(h)(X), Ad(h)(Y) \rangle = \int_G \langle Ad(gh)(X), Ad(gh)(Y) \rangle_0 \, d\mu(g)$$

$\tilde{g} = gh \Rightarrow g = \tilde{g} h^{-1}$

$$= \langle X, Y \rangle.$$

Hence, on $\mathfrak{g}$ $\exists$ $\langle\ ,\ \rangle$ bilinear, positive definite & symmetric

And $\langle ad_X Y, Z \rangle + \langle Y, ad_X Z \rangle = 0$ (since $\langle Ad(\exp(tX))(Y), Ad(\exp(tX))(Z) \rangle = \langle Y, Z \rangle$)

$\Rightarrow$ ad_x is skew symmetric in $\langle\ ,\ \rangle$

$$\Rightarrow \quad B(x, y) = tr(ad_x\, ad_y)$$

$$\Rightarrow \quad B(x, x) = -\, tr(\underbrace{ad_x\ (ad_x)^{tr}})$$

$$= -\sum A_{ij}^2 \quad = 0 \quad \text{iff } ad_x = 0$$

Namely $-B(x,y)$ is a Ad-invariant $\boxed{\text{product}}$ on $\mathfrak{g}$.
inner

$Rad(B) = \{x \mid ad_x = 0\} = \mathfrak{z}(\mathfrak{g})$.

Note on $\mathfrak{g}$ $\exists \langle\ ,\ \rangle$ positive definite bilinear form

(*) $\mathfrak{z}(\mathfrak{g})^{\perp} = \mathfrak{g}'$

$\langle x, [y,z]\rangle = \langle [x,y], z\rangle$, $\Rightarrow$ LHS $=0$ iff RHS $=0$ $\forall y\ z$.

$x \in \mathfrak{z}(\mathfrak{g})$, $\Leftrightarrow$ $x \in (\mathfrak{g}')^{\perp}$

Hence $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{g}'$. on $\mathfrak{g}'$, $B<0 \Rightarrow$ $\mathfrak{g}'$ is Semi-simple.

For (ii), if $\mathfrak{g}$ is simple & $B<0 \Rightarrow \mathfrak{g} = \mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$

$\mathfrak{g}$ is compact is due to the observations (a) $\mathfrak{z}(\mathfrak{g}) = 0$

$\Rightarrow$ $ad: \mathfrak{g} \to Aut(\mathfrak{g})$ is 1-1 into $\sim$ image of Ad.

Namely $ad: \mathfrak{g} \cong int(\mathfrak{g}) = Lie(Int(\mathfrak{g}))$

(b) and $Int(\mathfrak{g}) \subset O(\mathfrak{g})$ — must be compact.

Since $A^{\in}$ preserves $-B$.

Since $\mathfrak{g}$ can be equipped with ad-invariant metric $-B$.

(ii)

$(\Rightarrow)$ is trivial by (i). since $\mathfrak{g}$ is simple & $\mathfrak{z}(\mathfrak{g}) = 0$

$\Rightarrow B<0$

Corollary: $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ & $\mathfrak{g}_i$ are simple $B|_{\mathfrak{g}_i} < 0$

Summary: $\mathfrak{g}$ is Compact iff $\exists \langle\ ,\ \rangle$ Product which is ad invariant $\underbrace{\langle ad_x y, z\rangle + \langle y, ad_x z\rangle}_{=0}$

A Semi-simple $\mathfrak{g}$ is compact iff $-B > 0$.

2 Weyl theorem: (i) A compact Lie group with finite center must be semi-simple

(ii) Any Lie Group G has a Lie algebra which is Semi-Simple & $-B>0$ must be compact. It has finite center & $|\pi_1(G)|<+\infty$

In particular, the universal cover $\tilde{G}$ is compact & has finite center.

Pf: (i) The proof here uses Riemannian geometry, following Milnor.
(ii) $\exists$ Algebraic proof. See Varadarajan §4.11.
(iii) Altenatively, Let G_1 be the compact one with $Lie(G_1)=\mathfrak{g}$. (existence is by the above discussion)
$\tilde{G}_1$ its universal cover. If $\pi_1(G_1)$ is finite, $\Rightarrow$ $\tilde{G}$ is compact $\Rightarrow$ Any G with $Lie(G)=\mathfrak{g}$ is compact.
Bochner technique shows $b_1(G_1)=0$, Hence $\pi_1(G_1)$ is finite.

Defn: G is called $\begin{cases} \text{nilpotent} \\ \text{solvable} \\ \text{semi-simple} \end{cases}$ if $\mathfrak{g}$ is $\begin{cases} \text{—} \\ \text{—} \\ \text{—} \end{cases}$

(i) It is easy since $Z(G)$ is finite $\Rightarrow$ $\mathfrak{z}(\mathfrak{g})=0$ $\Rightarrow$ $\mathfrak{g}$ is Semi simple.
$(Z(G))_0=\{e\}$.

[$Z(G)$ has Lie algebra $\mathfrak{z}(\mathfrak{g})$. — an exercise problem.

Key is $\boxed{\varphi(\exp(tX)) = \exp(t\, d\varphi(X))}$

Apply it to $\varphi = Ad$

$\Rightarrow$ $Ad(\exp(tX)) = \exp(t\, ad_X) = id$

if $x\in \mathfrak{z}(\mathfrak{g}) \Rightarrow$ $Ad(\exp(tX)) = id.$

Apply it to a

$$\underset{\exp(tX)}{a}\left(\exp(sY)\right) = \exp\left(s\, \underbrace{Ad(\exp(tX))}_{=id}(Y)\right)$$

$\Rightarrow$ $\exp(tX)\left(\exp(sY)\right)\exp(-tX) = \exp(sY)$

$\Rightarrow$ For local g $\exp(tX)\, g = g\, \exp(-tX)$

$\Rightarrow$ $\exp(tX) \in (Z(G))_0$]

(ii) We can prove it using <u>Bonnett – Meyer theorem</u>.

(M, g), a complete Riemannian manifold, with $\mathrm{Ric} \geq \delta g$ $\Updownarrow$ $\mathrm{Ric}(X, Y) \geq \delta\, g(X, Y)$, $\delta > 0$

$\Rightarrow$ M is compact & $|\pi_1(M)| < +\infty$

$\tilde{G}$ has Lie algebra $\mathfrak{g}$ $\quad B < 0$ on $\mathfrak{g}$.

$\Rightarrow$ B is Ad invariant (Since B is <u>ad-invariant</u>)

$\gamma: \mathbb{R} \to G$, $t \mapsto \gamma(t)$

(a) $\Rightarrow$ B can be extended to $\tilde{G}$ as a bi-invariant Riemannian metric

$$\boxed{\begin{aligned}&\langle (dR_g)_e(X), (dR_g)_e(Y)\rangle_g \\ &= \langle dL_{g^{-1}}(dR_g)(X),\ dL_{g^{-1}} dR_g(Y)\rangle_e \\ &= \langle Ad(g^{-1})(X),\ Ad(g^{-1})(Y)\rangle \\ &= \langle X, Y\rangle_e\end{aligned}}$$

Connection or Co-variant derivative.

(b) $\nabla_X Y = \frac{1}{2}[X, Y]$. $\quad X, Y$ left invariant

— use Koszul formula

(c) $R(X, Y)Z = -\frac{1}{4}[[X, Y], Z]$

$R(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$

(d) $\mathrm{Ric}(X, X) = \sum_i \langle R(e_i, X)X, e_i\rangle = -\frac{1}{4}\langle [[e_i, X], X], e_i\rangle$ $\quad \{e_i\}$

Milnor: "Morse theory"

$= -\frac{1}{4} B(X, X)$ $\quad \to \mathrm{tr}\ ad_X\, ad_X$

$\Rightarrow$ $(\tilde{G}, -B)$, $\quad -B = g$. $\quad \mathrm{Ric} = \frac{1}{4}(-B)$ $\quad$ Hence Meyer theorem applies.

(3). Abelian, nilpotent, solvable notions are all from the corresponding notions for groups.

G. $\quad xy = yx \iff \exp(tX)\exp(sY) = \exp(sY)\exp(tX)$

$\iff \exp(tX)\exp(sY)\exp(-tX) = \exp(sY)$

$\iff Ad_{\exp(tX)}(Y) = Y \iff Ad_{\exp(tX)} \equiv id$

$\Leftrightarrow$ $ad_x = 0$ namely $[x, y] = 0$ $\forall x, y$.

G is called solvable if $G_1 = [G, G]$

$[G, G] =$ the subgroup generated by $aba^{-1}b^{-1}$. $a, b \in G$

$G_i := [G_{i-1}, G_{i-1}]$ & $G_k = \{e\}$ for some k.

Since $\frac{\partial^2}{\partial t \partial s}\Big|_{\substack{s=0\\t=0}} \left(\exp(tX)\exp(sY)\exp(-tX)\exp(-sY) \right)$

$$= \frac{\partial}{\partial t}\Big|_{t=0} \left[Ad(\exp(tX))(Y) - Y \right] = ad_X Y = [X, Y]$$

The construction is related.

" G is solvable iff $\mathfrak{g}$ is solvable "

$\uparrow$ Connected Lie group

Ideal $\longleftrightarrow$ normal subgroup

sub-algebra $\longleftrightarrow$ Lie subgroups

$[x, y] \in h$ $\quad$ $\boxed{ghg^{-1} \in H}$

$\forall x \in h, y \in \mathfrak{g}$ $\quad$ $\Updownarrow$

$Ad(\exp(ty))(x) \in h$

$\Leftrightarrow$ $[y, x] \in h$ $\forall y \in \mathfrak{g}$.

We shall see that for semi-simple Lie algebra, one can learn a lot by looking the maximum Abelian sub-algebr.

$\mathfrak{t} \subset \mathfrak{g}$ is called a maximum Abelian subalgebra if $\forall \mathfrak{t}'$ - Abelian subalgebra, $\mathfrak{t} \subset \mathfrak{t}' \Rightarrow \mathfrak{t} = \mathfrak{t}'$

$T \subset G$ is called a maximum Torus if $\forall T'$ - a compact Abelian subgroup $\subset G$

$T \subset T' \Rightarrow T = T'$

④ Normalizer & Centralizer

G, $\boxed{K \text{ subgroup}}$ $\qquad N(K) := \{ g \mid gKg^{-1} \subset K \}$

a group since K is

$\mathfrak{g}$. $\boxed{\mathfrak{h} \text{ subalgebra}}$ $\qquad N(\mathfrak{h}) := \{ a \in \mathfrak{g} \mid [a, \mathfrak{h}] \subset \mathfrak{h} \}$

$$[[x, y], z] = -[\underbrace{[y, z]}_{\in \mathfrak{h}}, x] - [\underbrace{[z, x]}_{\in \mathfrak{h}}, y] \in \mathfrak{h} \quad \forall z \in \mathfrak{h}.$$

$N(K)$ is the maximal subgroup L such that K is normal in L.

$N(\mathfrak{h})$ is the maximal subalgebra $\mathfrak{k}$ such that $\mathfrak{h}$ is an ideal in $\mathfrak{k}$.

Centralizer: G, $\boxed{K - \text{any set}}$

$$Z(K) := \{ g \mid gx = xg \ \ \forall x \in K \}$$

$g^{-1} \in Z(K)$ if $g \in Z(K)$.

any set η

$Z(\eta) := \{ x \mid [x, h] = 0 \ \forall h \in \eta \}$

η can be a set

$[[x\ y]\ h]$

$= -[\underbrace{[y, h]}_{0}, x] - [\underbrace{[h, x]}_{0}, y]$

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⑤ $Int(\mathfrak{g}) \subset (Aut(\mathfrak{g}))_0$ by the definition.

$i : Int(\mathfrak{g}) \hookrightarrow (Aut(\mathfrak{g}))_0$ is the inclusion map

Since $di : int(\mathfrak{g}) \xrightarrow{= ad_{\mathfrak{g}}} Der(\mathfrak{g})$ is an isomorphism, by what we proved i is a covering map which is onto.

(Also 1-1 since it is the inclusion map).

$\Rightarrow \quad Aut(\mathfrak{g}) / Int(\mathfrak{g}) \simeq Aut(\mathfrak{g}) / (Aut(\mathfrak{g}))_0$ — hence discrete.